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Sharpening Jordan's inequality and the Yang Le inequality

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Abstract

In this work, the following inequality:

$$\frac{\sin x}{x} \le \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \qquad x \in (0, \pi/2]$$

is established. An application of this inequality gives an improvement of the Yang Le inequality [C.J. Zhao, Generalization and strengthening of the Yang Le inequality, Math. Practice Theory 30 (4) (2000) 493–497 (in Chinese)]:

$$(n-1)\sum_{k=1}^{n} \cos^2 \lambda A_k - 2\cos \lambda\pi \sum_{1 \le i < j \le n} \cos \lambda A_i \cos \lambda A_j \le 4\binom{n}{2}\left(\lambda^3 + \frac{\lambda(1-\lambda^2)}{2}\pi\right)^2,$$

where $A_i > 0$ $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} A_i \le \pi$, $0 \le \lambda \le 1$, and $n \ge 2$ is a natural number.

Keywords: Upper bound; Jordan's inequality; Yang Le inequality

1. Introduction

The following theorem is known as Jordan's inequality [1]:

Theorem 1. *If* $0 < x \le \pi/2$, *then*

$$\frac{\sin x}{x} \ge \frac{2}{\pi} \tag{1}$$

with equality if and only if $x = \pi/2$.

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Debnath and Zhao [2] have obtained a new lower bound for the function $\frac{\sin x}{x}$. Their result reads as follows:

Theorem 2. *If* $0 < x \le \pi/2$, *then*

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \tag{2}$$

with the equality if and only if $x = \pi/2$.

In this work, we give an upper bound for $\frac{\sin x}{x}$. We have:

Theorem 3. *If* $0 < x \le \pi/2$, *then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \le \frac{\sin x}{x} \le \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2) \tag{3}$$

with the equalities if and only if $x = \pi/2$. *Furthermore,* $\frac{1}{\pi^3}$ *and* $\frac{\pi-2}{\pi^3}$ *are the best constants in* (3).

In [2] the authors have obtained an improvement of the Yang Le inequality.

Theorem 4. *Let* $A_i > 0$ $(i = 1, 2, \ldots, n)$ *with* $\sum_{i=1}^{n} A_i \le \pi$, *let* $0 \le \lambda \le 1$ *and let* $n \ge 2$ *be a natural number. Then*

$$\binom{n}{2} \lambda^2 (3 - \lambda^2)^2 \cos^2 \frac{\lambda}{2}\pi \le (n-1) \sum_{k=1}^{n} \cos^2 \lambda A_k - 2 \cos \lambda\pi \sum_{1 \le i < j \le n} \cos \lambda A_i \cos \lambda A_j \le \binom{n}{2} \lambda^2 \pi^2. \tag{4}$$

Using the right inequality in (3), we shall establish and prove the following improvement of the Yang Le inequality.

Theorem 5. *Let* $A_i > 0$ $(i = 1, 2, \ldots, n)$ *with* $\sum_{i=1}^{n} A_i \le \pi$, *let* $0 \le \lambda \le 1$ *and let* $n \ge 2$ *be a natural number. Then*

$$(n-1) \sum_{k=1}^{n} \cos^2 \lambda A_k - 2 \cos \lambda\pi \sum_{1 \le i < j \le n} \cos \lambda A_i \cos \lambda A_j \le 4 \binom{n}{2} \left(\lambda^3 + \frac{\lambda(1 - \lambda^2)}{2}\pi \right)^2. \tag{5}$$

2. A lemma

Lemma 1 (*[3,5]*). *Let* $f, g : [a, b] \to R$ *be two continuous functions which are differentiable on* (a, b). *Further, let* $g' \ne 0$ *on* (a, b). *If* f'/g' *is decreasing on* (a, b), *then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

and

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

are also decreasing on (a, b).

3. A short proof of Theorem 3

Let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -4x^2$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^3$, and $x \in (0, \pi/2]$. Then we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{8}\frac{\sin x - x\cos x}{x^3} = \frac{1}{8}\frac{f_3(x)}{f_4(x)},$$
$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{3}\frac{\sin x}{x} = \frac{1}{3} f_1(x).$$

Since $f_1(x) = \frac{\sin x}{x}$ is decreasing on $(0, \pi/2)$ or $\frac{f_3'(x)}{f_4'(x)}$ is decreasing on $(0, \pi/2)$, $\frac{f_3(x)}{f_4(x)} = \frac{f_3(x)-f_3(0)}{f_4(x)-f_4(0)}$ is decreasing on $(0, \pi/2)$ by Lemma 1. So $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on $(0, \pi/2)$, and $h(x) = \frac{\frac{\sin x}{x} - \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}}{\pi^2 - 4x^2} = \frac{f_1(x)-f_1(\frac{\pi}{2})}{f_2(x)-f_2(\frac{\pi}{2})}$ is decreasing on $(0, \pi/2)$ by Lemma 1.

Furthermore, $\lim_{x\to 0^+} h(x) = \frac{\pi-2}{\pi^3}$ and $\lim_{x\to\frac{\pi}{2}^-} h(x) = \frac{1}{\pi^3}$; thus $\frac{\pi-2}{\pi^3}$ and $\frac{1}{\pi^3}$ are the best constants in (3).

4. The proof of Theorem 5

Let $H_{ij} = \cos^2\lambda A_i + \cos^2\lambda A_j - 2\cos\lambda\pi\cos\lambda A_i\cos\lambda A_j$. It follows from [4] that

$$H_{ij} \le 4\sin^2\frac{\lambda}{2}\pi. \tag{6}$$

Let $1 \le i < j \le n$. Taking the sum for all inequalities in (6), we obtain

$$\sum_{1\le i<j\le n} H_{ij} \le \sum_{1\le i<j\le n} 4\sin^2\frac{\lambda}{2}\pi. \tag{7}$$

It follows from the definition of H_{ij} that

$$\begin{aligned}\sum_{1\le i<j\le n} H_{ij} &= \sum_{1\le i<j\le n} (\cos^2\lambda A_i + \cos^2\lambda A_j - 2\cos\lambda\pi\cos\lambda A_i\cos\lambda A_j)\\ &= (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi\sum_{1\le i<j\le n}\cos\lambda A_i\cos\lambda A_j.\end{aligned} \tag{8}$$

Making use of the second inequality in (3) we obtain

$$\begin{aligned}\sum_{1\le i<j\le n} 4\sin^2\frac{\lambda}{2}\pi &\le \sum_{1\le i<j\le n} 4\left(\frac{\lambda}{2}\pi - \frac{4(\pi-2)}{\pi^3}\left(\frac{\lambda}{2}\pi\right)^3\right)^2\\ &= 4\binom{n}{2}\left(\lambda^3 + \frac{\lambda(1-\lambda^2)}{2}\pi\right)^2.\end{aligned} \tag{9}$$

Substituting (8) and (9) into (7), we obtain (5).

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